Division **M**

Mathematical Olympiads

November 14, 2018

for Elementary & Middle Schools

Contest **1**

Directions to Students: After all questions have been read by your PICO, you will have 30 minutes to complete this contest. You may not have a pen or pencil in your hand while the PICO reads the set of questions to the class. Calculators are not permitted. All work is to be done on the pages provided. No additional scrap paper is to be used. Answers must be placed in the corresponding boxes in the answer column.

Name: ______________________________

1A Compute the simplified numeric value of $(16 \div 8 \times 4)$ subtracted from $(16 \times 8 \div 4)$.

1B XYX and YXY represent two 3-digit whole numbers in which X and Y are distinct non-zero digits. Find the greatest possible sum of XYX and YXY.

1C Two pairs of prime numbers, P and Q, have the relationship that $3P + 5Q = 121$. Find the two possible values of $P + Q$.

Name: ______________________________

Answer Column
1A
1B
1C
1D
1E
Do Not Write in this Space. For PICO's Use Only. *SCORE:*

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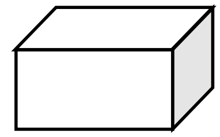
1D Three of the faces of a rectangular box have areas of 30 square inches, 18 square inches, and 15 square inches. How many cubic inches is the volume of the box?



1E Betsy collects sports posters. Exactly one of the statements below is false.

Betsy has at least 6 sports posters.
Betsy has at least 15 sports posters.
Betsy has at least 19 sports posters.
Betsy has at least 23 sports posters.

What is the greatest possible number of posters in Betsy's collection?

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Division **M**

Mathematical Olympiads
December 12, 2018
for Elementary & Middle Schools

Contest **2**

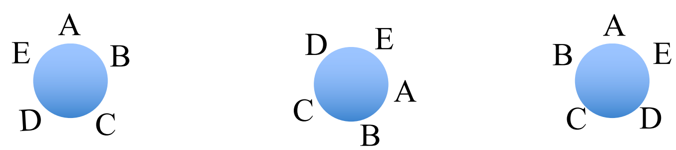
Directions to Students: After all questions have been read by your PICO, you will have 30 minutes to complete this contest. You may not have a pen or pencil in your hand while the PICO reads the set of questions to the class. Calculators are not permitted. All work is to be done on the pages provided. No additional scrap paper is to be used. Answers must be placed in the corresponding boxes in the answer column.

Name: ____

2A Compute the whole number value of $\frac{7982}{0.7982} - 7982$.

2B Mr. and Mrs. Hansen and their three children, are to be seated about a circular table. In how many different ways can the family be seated if Mr. and Mrs. Hansen are seated next to one another?

Case 1 and 2 are not considered different (one is a rotation of the other). Case 3 is different from the first two cases.



Case 1 Case 2 Case 3

2C Starting at midnight, Aarf the Dog barks for 15 seconds and then is silent for the next 25 seconds. Aarf the Dog continues this bark-fest until 1:01 AM that same morning. How many times did Aarf the Dog bark for 15 seconds?

Name: ______________________________

Answer Column
2A
2B
2C
2D
2E
Do Not Write in this Space. For PICO's Use Only. *SCORE:*

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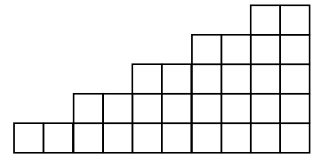
2D Thirty unit squares are arranged, sharing common sides and vertices, in the "step" arrangement as shown. How many squares with whole number length sides can be found in this arrangement?



2E XYX and YXY represent two 3-digit whole numbers in which X and Y are distinct non-zero digits. How many different values are possible for the sum XYX + YXY?

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Division **M**

Mathematical Olympiads

January 16, 2019

for Elementary & Middle Schools

Contest **3**

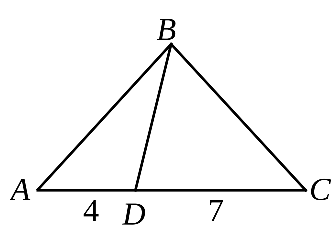
Directions to Students: After all questions have been read by your PICO, you will have 30 minutes to complete this contest. You may not have a pen or pencil in your hand while the PICO reads the set of questions to the class. Calculators are not permitted. All work is to be done on the pages provided. No additional scrap paper is to be used. Answers must be placed in the corresponding boxes in the answer column.

Name: ____

3A Simplify: $\sqrt{(2^3)^4}$

3B Simplify: $(1 + 2 - 3) + (4 + 5 - 6) + (7 + 8 - 9) + \ldots + (94 + 95 - 96) + (97 + 98 - 99)$.

3C Given ΔABC with point D on side $\overline{AC}$ such that $AD = 4$ and $CD = 7$. If the area of ΔABD is 10 square units, find the number of square units in the area of ΔABC.



Name: ______________________________

Answer Column
3A
3B
3C
3D
3E
Do Not Write in this Space. For PICO's Use Only. *SCORE:*

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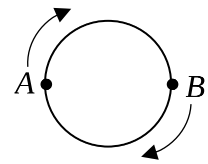
3D Andy and Brett begin at opposite points on a 300-foot circular track, and jog about the track in the directions shown. Andy jogs at 4.3 feet/sec and Brett jogs at 5.5 feet/sec. How many seconds will it take for Brett and Andy to meet for the first time?



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3E Some 3-digit whole numbers have repeated digits. Two examples are 122 and 777. Exactly K% of all 3-digit whole numbers have repeated digits, where K is a whole number. Find K.

Division
M

Mathematical Olympiads
February 13, 2019
for Elementary & Middle Schools

Contest
4

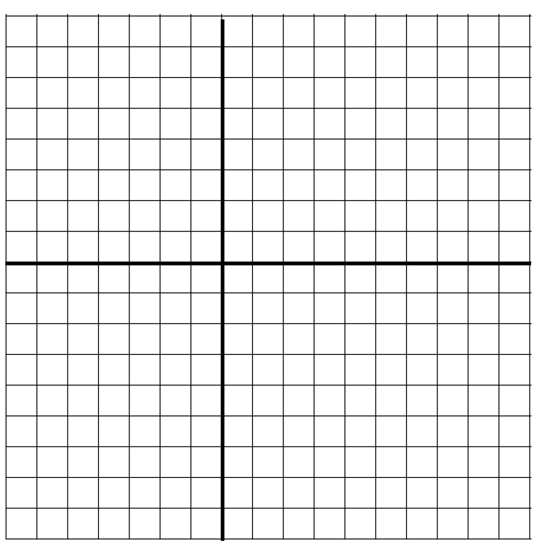
<u>Directions to Students</u>: After all questions have been read by your PICO, you will have 30 minutes to complete this contest. You may not have a pen or pencil in your hand while the PICO reads the set of questions to the class. Calculators are not permitted. All work is to be done on the pages provided. No additional scrap paper is to be used. Answers must be placed in the corresponding boxes in the answer column.

Name: ____

4A By how much does $\sqrt{0.25}$ exceed 0.25^2?

4B Find the greatest numeric value of $\frac{x-y}{x}$ given that x is a whole number factor of 20 and y is a whole number factor of 12. Express the answer as a fraction in simplest form.

4C How many points (x, y) on the line segment $\overline{AB}$ with $A(-6, 5)$ and $B(10, -7)$ have integer values for both coordinates x and y?



Name: ______________________________

Answer Column
4A
4B
4C
4D
4E
Do Not Write in this Space. For PICO's Use Only. *SCORE:*

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4D In the 5-digit whole number XYYYX, X and Y represent different non-zero digits. If XYYYX is divisible by 11, how many different values of XYYYX are possible?

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4E In algebra, we learn that $(a+b)^2 = a^2 + 2ab + b^2$. Use this information to help compute the whole number value of the expression: $\dfrac{2019^2 - 2000^2 - 19^2}{2000 \times 19}$.

Division **M**

Mathematical Olympiads

March 6, 2019

for Elementary & Middle Schools

Contest **5**

Directions to Students: After all questions have been read by your PICO, you will have 30 minutes to complete this contest. You may not have a pen or pencil in your hand while the PICO reads the set of questions to the class. Calculators are not permitted. All work is to be done on the pages provided. No additional scrap paper is to be used. Answers must be placed in the corresponding boxes in the answer column.

Name: ____

5A Compute the integer that is closest in numerical value to the expression $\sqrt{\frac{33}{0.499} - 17.1}$

5B Of all the negative quotients found by dividing two distinct integers chosen from the set {–7, –5, –4, –3, –2, 0, 1, 2, 6}, which is the greatest in value?

5C What is the greatest number of consecutive integers that sum to –11?

Name: ______________________________

Answer Column
5A
5B
5C
5D
5E
Do Not Write in this Space. For PICO's Use Only. *SCORE:*

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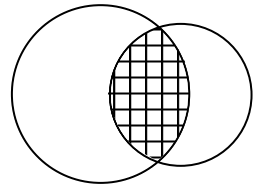
5D A circle of radius 4 inches and a circle of radius 3 inches overlap in the crosshatched region as shown. The area of the crosshatched region is 6π square inches. Find the number of square inches in the total area of the two regions without any crosshatching. Express the answer in terms of π.



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5E The compound inequality $a<b<c$ means that $a<b$ and at the same time $b<c$. For example, $1<x<5$ has integer solutions 2, 3, and 4.
Find the sum of all the integer values of N that satisfy the compound inequality $2 < |N| < 6$.

Division M | Mathematical Olympiads November 14, 2018 for Elementary & Middle Schools | Contest 1

SOLUTIONS AND ANSWERS

1A	1B	1C	1D	1E
24	1887	25 & 39	90	22

1A **METHOD 1** *Strategy: Use the order of operations.*
$(16 \times 8 \div 4) - (16 \div 8 \times 4) = (128 \div 4) - (2 \times 4) = 32 - 8 =$ **24**.

METHOD 2 *Strategy: Rewrite as a fraction and a product.*
$\left(16 \times \frac{8}{4}\right) - \left(\frac{16}{8} \times 4\right) = (16 \times 2) - (2 \times 4) = 32 - 8 = 24.$

FOLLOW UP: Compute: $(75 \times 5 \div 3) + (75 \div 5 \times 3)$. [170]

1B *Strategy: Determine the greatest possible sum for X + Y.*
The greatest value for the sum of X and Y occurs when one letter equals 8 and the other letter equals 9. Therefore, the greatest possible sum is 898 + 989 = 989 + 898 = **1887**.

FOLLOW UPS: (1) What is the least possible absolute value for XYX – YXY? [91]
(2) Let ABC and CBA represent 3-digit whole numbers. If A, B, and C are distinct non-zero digits find the least possible value of ABC + CBA. [363]

1C **METHOD 1** *Strategy: List the possible primes and then use number sense.*
Since P and Q are each prime and $3P + 5Q = 121$, the greatest prime possible is 37 since $3 \times 37 = 111$. The list of possible primes is: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, and 37. Rearrange the equation into $5Q = 121 - 3P$. Hence, $121 - 3P$ is a multiple of 5 so $3P$ must have a units digit of either 6 or 1.

If $P = 2$, $3P$ has a units digit of 6. Thus $5Q = 121 - 3(2) = 115$ and $Q = 23$. Thus one solution is $P = 2$ and $Q = 23$.

If $P = 7$, 17 and 37, $3P$ has a units digit of 1.
When $5Q = 121 - 3(7) = 100$, $Q = 20$ which is NOT prime.
When $5Q = 121 - 3(17) = 70$, $Q = 14$ which is NOT prime.
When $5Q = 121 - 3(37) = 10$, $Q = 2$ which is prime so we have a second solution.

The two possible values for $P + Q$ are 2 + 23 = **25** and 37 + 2 = **39**.

METHOD 2 *Strategy: Use your knowledge of prime numbers.*
Since 121 is odd: either ($3P$ is even and $5Q$ is odd) or ($3P$ is odd and $5Q$ is even).
Case I: ($3P$ is even and $5Q$ is odd). Let $P = 2$ so $3P = 6$, $5Q = 115$ and $Q = 23$.
It follows that $P + Q = 2 + 23 = 25$.
Case II: ($3P$ is odd and $5Q$ is even). Let $Q = 2$ so $5Q = 10$, $3P = 111$ and $P = 37$.
It follows that $P + Q = 2 + 37 = 39$.
Therefore, the two possible values for $P + Q$ are 25 and 39.

FOLLOW UP: Given that P, Q, and R are all prime numbers, what is the value of P if P + Q + R = 65 and 2Q + 3R = 53? [41]

1D **METHOD 1** *Strategy: Use the formulas for area and volume of a rectangular prism.*

Let L = length, W = width and H = height. The areas of three of the faces are L × W = 30, L × H = 18, and W × H = 15. Multiply the three face areas to get (L × W) × (L × H) × (W × H) = $L^2 \times W^2 \times H^2$ = $(L \times W \times H)^2$ = 30 × 18 × 15 = 15 × 2 × 2 × 3 × 3 × 15.
The volume of the box is 15 × 2 × 3 = **90**.

METHOD 2 *Strategy: Find common factors for the given areas.*

Since 30 = 6 × 5, 18 = 6 × 3 and 15 = 5 × 3, the length can be 6, the width can be 5 and the height can be 3. Therefore, the volume is 6 × 5 × 3 = 90.

FOLLOW UPS: (1) Find the length of a diagonal of the original rectangular box. [$\sqrt{70}$] *(2) A box has a volume of 70 cubic inches. If the length, width, and the height of the box are all different prime numbers, what is the surface area of the box?* [118 square inches]

1E **METHOD 1** *Strategy: Use logical reasoning.*

If Betsy has 6 posters, she does not have at least 15, 19, or 23 posters. If she has 22 posters all but the last statement will be true. If she has 18 posters, two sentences are false. To make only one of the statements false, she needs to have **22** posters. The first three statements will be true and the last statement will be false.

METHOD 2 *Strategy: Apply logic.*

The last statement must be the only false statement, since any previous false statement would make all later statements false as well. Betsy has at least 19 sports posters, but not 23 or more posters. Therefore, Betsy has at most 22 posters.

FOLLOW UP: The sum of the number of pieces of candy that Joe has and the number of pieces of candy that Tracy has is 26. Joe gives Tracy n pieces of candy so that she will have twice the number of pieces of candy as Joe initially had. How many different values are there for n? [Assume no piece of candy can be split apart.] [5]

__NOTE:__ Other FOLLOW UP problems related to some of the above can be found in our three contest problem books and in "Creative Problem Solving in School Mathematics."
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Division **M**

Mathematical Olympiads

December 12, 2018

for Elementary & Middle Schools

Contest **2**

SOLUTIONS AND ANSWERS

2A **METHOD 1** *Strategy: Use knowledge of fractions and decimals.*

Change 0.7982 into the equivalent fraction $\frac{7982}{10000}$. It follows that

$7982 \div \frac{7982}{10000} = 7982 \times \frac{10000}{7982} = 10000$. Thus 10000 – 7982 = **2018**.

METHOD 2 *Strategy: Use a pattern.*

Notice that $\frac{7}{0.7} = 10$, $\frac{79}{0.79} = 100$, $\frac{798}{0.798} = 1000$ and $\frac{7982}{0.7982} = 10000$. It follows that 10000 – 7982 = 2018.

METHOD 3 *Strategy: Use factoring.*

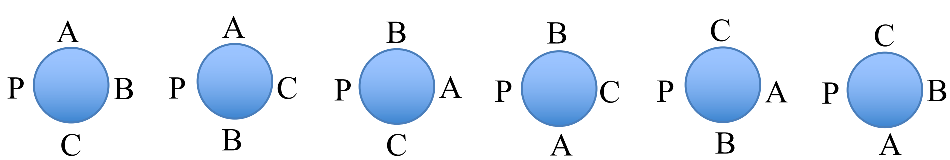
$$\frac{7982}{0.7982} - 7982 = 7982\left(\frac{1}{0.7982} - 1\right) = 7982\left(\frac{1}{0.7982} - \frac{0.7982}{0.7982}\right) = 7982\left(\frac{0.2018}{0.7982}\right)$$

Multiply numerator and denominator of the fraction by 10000/10000 and then convert 7982/7982 to the number 1 to get 2018.

FOLLOW UP: Compute $\frac{7982}{0.7982} + \frac{0.7982}{7982}$. [10000.0001]

2B **METHOD 1** *Strategy: Consider the parents as a single unit.*

Let P represent the parents and let A, B, and C be the three children. Consider the circular arrangements of the 4 letters. There are only 6 cases.



Let M represent mom and D represent dad. We can replace each letter P with either DM or MD. Therefore there are actually **12** different ways to sit at the table.

METHOD 2 *Strategy: Apply the formula for circular permutations.*

The number of ways to permute *n* people around a circular table is $(n - 1)!$ where the symbol "!" means factorial. Grouping the parents as a single unit means that there are (4 – 1)! = 3! = 6 ways to arrange 4 people about a round table. Double this number since mom and dad can sit in 2 different positions for each case.

FOLLOW UP: In the original problem what is the probability that children A and B are seated next to one another? [2/3]

2A	**2018**
2B	**12**
2C	**92**
2D	**53**
2E	**15**

2C **METHOD 1** _Strategy_: *Use the length of each bark-silent cycle.*

Since 15 seconds (barking) + 25 seconds (silent) = 40 seconds = 2/3 minute, in 60 minutes there will be 90 barking sessions (60 ÷ 2/3). For the last minute there will be 2 additional barking sessions. Thus, there will be 90 + 2 = **92** barking times from midnight until 1:01 AM.

METHOD 2 _Strategy_: *Find a pattern.*

Every 2 minutes Aarf barks 3 times, as illustrated below for the first 2 minutes after midnight:

12:00:00 – 12:00:15 barks	12:00:15 – 12:00:40 silent
12:00:40 – 12:00:55 barks	12:00:55 – 12:01:20 silent
12:01:20 – 12:01:35 barks	12:01:35 – 12:02:00 silent

It follows that in 60 minutes Aarf would bark 90 times [3 barks per 2 minutes times 30]. From 1:00 AM until 1:01 AM Aarf would bark 2 more times (see first minute above) and 90 + 2 = 92.

METHOD 3 _Strategy_: *Convert minutes into seconds.*

Change 61 minutes into 3660 seconds. Divide by 40, 3660/40 = 91 cycles and 20 seconds. Since 20 seconds allows for 1 additional bark cycle, 91 + 1 = 92.

FOLLOW UP: A dog barks every b minutes and a cat meows every m minutes. If b and m are prime numbers such that 1 ≤ b ≤ 10 and 10 ≤ m ≤ 20, what is the greatest number of times they can bark and meow at the same time in a 2-hour period? [5]

2D _Strategy_: *Consider the number of squares based on side length.*

Side Length	1	2	3	4
Number of Squares	30	16	6	1

The total number of squares with whole number sides is 30 + 16 + 6 + 1 = **53**.

FOLLOW UPS: (1) Using the same arrangement of squares, find the maximum number of squares with only even number side lengths if there is no overlap. [6] *(2) What is the perimeter of the given shape?* [30]

2E **METHOD 1** _Strategy_: *Apply logical reasoning.*

When adding XYX + YXY, notice that the sum of the digits in each place can be represented by X + Y. Since we are looking for different values for the possible sums, look for all possible values for X + Y.

For example, examine sums that add to 5: 141 + 414 = 555 and 232 + 323 = 555.
The least sum for X + Y occurs when the two numbers are 1 and 2. The sum is 3.
The greatest sum for X + Y occurs when the two numbers are 8 and 9. The sum is 17.
There are 17 – 3 + 1 = **15** different sums between and including 3 and 17.

METHOD 2 _Strategy_: *Find unique sums.*

Notice that XYX + YXY = (X + Y) hundreds + (Y + X) tens + (X + Y) ones. Notice that it doesn't matter whether X or Y is the greater digit, since the total for each place is their sum and addition is commutative. If the greater digit is 9 there are 8 sums possible for each place (a regrouping will result when completing the final sum) 9 + 8 = 17, 9 + 7 = 16, …, 9 + 1 = 10. If the greater digit is 8, there is only one new sum 8 + 1 = 9. Similarly, there is only one new sum if the greater digit is 7, 6, …, 2. Then 8 sums + 7 sums = 15 sums.

NOTE: *Other FOLLOW UP problems related to some of the above can be found in our three contest problem books and in "Creative Problem Solving in School Mathematics."*
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SOLUTIONS AND ANSWERS

3A	64
3B	1584
3C	27.5
3D	125
3E	28

3A **METHOD 1** _Strategy_: *Apply the definition of powers and roots.*

Since $(2^3)^4 = (2^3)(2^3)(2^3)(2^3) = 2^{12}$, $\sqrt{2^{12}} = \sqrt{(2^6)(2^6)} = 2^6 = \mathbf{64}$

METHOD 2 _Strategy_: *Apply the definition powers another way.*

Since $(2^3) = 8$, $\sqrt{8^4} = \sqrt{8 \times 8 \times 8 \times 8} = \sqrt{64 \times 64} = 64$

METHOD 3 _Strategy_: *Use trial and error.*

Calculate $(2^3)^4 = 4096$. So $\sqrt{(2^3)^4} = \sqrt{4096}$. We need to find the number which, when squared, equals 4096. $50^2 = 2500$, too small. $60^2 = 3600$, too small. $70^2 = 4900$, too large. $65^2 = 4225$, slightly too large. $64^2 = 4096$. $\sqrt{64^2} = 64$

3B **METHOD 1** _Strategy_: *Simplify each expression in the parentheses.*

The value of the numbers in the parentheses result in the following set of addends: 0 + 3 + 6 + 9 + 12 + ... + 93 + 96. Ignore the 0 and add the 3 and 96, which gives 99. Add 6 and 93, which also gives 99. Note that the sum of the next term from the beginning of the series and the previous term from the end of the series is always 99. Since 3 is the first term, 6 the second term, 9 the third term, and so on, 96 must be the 32nd term (96/3 = 32). Therefore, there will be 16 pairs of terms that add to 99 and 16 × 99 = **1584.**

METHOD 2 _Strategy_: *Simplify each expression and factor the common factor.*

As in Method 1, we get 3 + 6 + 9 + 12 + ... + 93 + 96. Factoring out the common factor gives 3(1 + 2 + 3 + 4 + ... + 31 + 32). To find this sum use the same rationale as in Method 1 or use the formula for finding the sum, S, of n consecutive counting numbers starting with 1.
The formula is $S = n(n + 1)/2 = 32(32 + 1)/2 = 528$. Multiply by 3 to get 1584.

FOLLOW UP: Find the sum of 1 – 2 + 3 – 4 + 5 – 6 + ... + 99 – 100. [–50]

3C **METHOD 1** _Strategy_: *Use the formula for the area of a triangle.*

The area of a triangle formula is A = (base × height)÷2 = $bh/2$. Apply this formula to find the height of ΔABD: $10 = 4h/2$ so $h = 5$. Since the height of ΔABC is the same as the height of ΔABD, the area of $\Delta ABC = (11 \times 5)/2 =$ **55/2** or **27.5**.

METHOD 2 _Strategy_: *Find the ratio of the bases of the triangles.*

If two triangles have the same height, the ratio of their areas equals the ratio of their bases. Therefore $\frac{A_{\Delta ABC}}{A_{\Delta ABD}} = \frac{AC}{AD}$ so $\frac{A_{\Delta ABC}}{10} = \frac{11}{4}$ and $A_{\Delta ABC} = 27.5$.

FOLLOW UP: Find the ratio of the area of ΔBCD to the area of ΔABD. [7:4]

3D **METHOD 1** <u>Strategy</u>: *Apply the formula rate × time = distance.*
Brett starts 300/2 = 150 feet behind Andy. Therefore, Brett has to travel 150 feet more than Andy in order for them to meet. Let t = the time it takes for Brett to meet up with Andy. Then $5.5t = 4.3t + 150$. It follows that $1.2t = 150$ so t = **125** seconds.

METHOD 2 <u>Strategy</u>: *Consider the difference in the rates that they jog.*
If we realize that Brett has to travel 150 feet more than Andy, assume that Andy does not move and that the rate that Brett jogs is 5.5 – 4.3 = 1.2 feet/second. Then $1.2t = 150$ and t = 125 seconds.

3E **METHOD 1** <u>Strategy</u>: *Find the percent of 3-digit whole numbers that have no repeated digits.*
If there are no repeated digits, then the first digit can be any of 9 possible digits (cannot be 0), the second digit can be any of the 9 remaining digits (can be 0), and the third digit can be any one of the 8 remaining digits. Therefore, there are 9 × 9 × 8 = 648 3-digit whole numbers that have no repeated digits. There are 9 × 10 × 10 = 900 3-digit whole numbers. The percent of 3-digit whole numbers that have no repeated digits is 648/900 = .72 = 72%. Therefore, 28% of 3-digit whole numbers have repeated digits so K = **28.**

METHOD 2 <u>Strategy</u>: *Find the number of possible 3-digit whole numbers with repeated digits.*
If two of the digits repeat, they would be of the form of ssd, sds, or dss.

The number of ssd types is 9 × 1 × 9 = 81 since the hundreds digit cannot be 0, the tens digit must be the same as the hundreds digit and the ones digit must be different from the first two digits.

The number of sds types is 9 × 9 × 1 = 81 since the hundreds digit cannot be 0, the tens digit can be 0 but must differ from the hundreds digit and the units digit must be the same as the hundreds digit.

The number of dss types is 9 × 9 × 1 = 81 since we can apply the reasoning from the first two cases.

The number of sss types is 9 × 1 × 1 = 9 since 000 is not a 3-digit number.

Therefore, there are 3 × 81 + 9 = 252 different 3-digit whole numbers with repeated digits. Since there are 9 × 10 × 10 = 900 3-digit whole numbers, the percent of 3-digit whole numbers with repeated digits is 252/900 = .28 = 28%. Therefore, K = 28.

FOLLOW UPS: (1) Find the percent of 4-digit whole numbers that have repeated digits. [49.6] *(1) Find the percent of 2-digit whole numbers that have repeated digits.* [10] *(3) Would the percent of 8-digit whole numbers that have repeated digits be greater than 50%?* [yes] *(4) Find the percent of 11-digit whole numbers that have repeated digits.* [100]

SOLUTIONS AND ANSWERS

4A	4B	4C	4D	4E
0.4375 or 7/16	19/20	5	8	2

4A **METHOD 1** *Strategy: Perform the indicated operations using decimals.*
The number $\sqrt{0.25} = 0.5$ and $0.25^2 = 0.0625$. Then $0.5000 - 0.0625 =$ **0.4375**.

METHOD 2 *Strategy: Perform the indicated operations using fractions.*
Since $0.25 = 1/4$, $\sqrt{0.25} = 1/2$ and $0.25^2 = 1/16$. Then $1/2 - 1/16 = 7/16$.

FOLLOW UP: How much greater is 2^5 than 2^{-5}? [31 31/32]

4B **METHOD 1** *Strategy: Factor each number, then maximize the given expression.*
The factors of 20 are 1, 2, 4, 5, 10, and 20. The factors of 12 are 1, 2, 3, 4, 6, and 12. Notice that for any of the values selected the numerator will always be less than the denominator since we are subtracting a positive number from x in the numerator. To maximize the numerator, let $y = 1$ and $x = 20$. The result in the numerator is 19, and the denominator is 20 making the fraction 19/20. This is only 1/20 away from 1. All other choices for x and y produce values less than 1 and further away from 1. Therefore, the greatest value for the fraction is **19/20**.

METHOD 2 *Strategy: Decompose the fraction.*
The given expression is equivalent to $\frac{x-y}{x} = \frac{x}{x} - \frac{y}{x} = 1 - \frac{y}{x}$. To maximize this value, minimize the fraction y/x. To accomplish this, find the least numerator and the greatest denominator. Therefore, $x = 20$ and $y = 1$, so the expression has the value 19/20.

FOLLOW UP: If x is randomly chosen from the positive factors of 20 and y is randomly chosen from the positive factors of 12, what is the probability that $\frac{x-y}{x}$ is negative? [7/18]

4C *Strategy: Use the slope of a line.*
The slope of the line is $\frac{\text{change in y}}{\text{change in x}} = \frac{-7-5}{10-(-6)} = \frac{-12}{16} = \frac{-3}{4}$. The points on the segment with integer coordinates can be found by starting at A and counting down 3 units and to the right 4 units (using the "rise" and the "run" of the slope). Therefore, the points on the segment with integer coordinates are $A(-6, 5)$, $P(-2, 2)$, $Q(2, -1)$, $R(6, -4)$, and $B(10, -7)$. The number of points on the segment with integer coordinates is **5**.

FOLLOW UP: Point M(x, y) is on the perpendicular bisector of $\overline{AB}$, and x and y are integers with $|x+y|$ as small as possible. What are the coordinates of M? [(2, –1)]

4D **METHOD 1** *Strategy: Apply the divisibility rule for 11.*
If XYYYX is divisible by 11, then X – Y + Y – Y + X is divisible by 11. This means that 2X – Y is divisible by 11. Since X and Y are single digit numbers, the only multiples of 11 that are possible values of 2X – Y are 0 and 11. Create a table of possible values for X and Y.

2X – Y	Possible (X, Y) values
0	(1, 2), (2, 4), (3, 6), (4, 8)
11	(6, 1), (7, 3), (8, 5), (9, 7)

Therefore, the number of different possible values of XYYYX that are multiples of 11 is **8**.

METHOD 2 *Strategy: Create a table of values for all possible X and Y values.*
Since the divisibility rule for 11 results in 2X – Y needing to be a multiple of 11, create a table for all possible values for X and Y.

Y \ X	1	2	3	4	5	6	7	8	9
1	1	3	5	7	9	**11**	13	15	17
2	**0**	2	4	6	8	10	12	14	16
3	-1	1	3	5	7	9	**11**	13	15
4	-2	**0**	2	4	6	8	10	12	14
5	-3	-1	1	3	5	7	9	**11**	13
6	-4	-2	**0**	2	4	6	8	10	12
7	-5	-3	-1	1	3	5	7	9	**11**
8	-6	-4	-2	**0**	2	4	6	8	10
9	-7	-5	-3	-1	1	3	5	7	9

Notice that the only multiples of 11 in the table are 0 and 11. Since there are 4 pairs of values that equal 0 and 4 pairs of values that equal 11, there are only 8 pairs of values for X and Y that result in XYYYX to be divisible by 11. These 8 numbers are: 12221, 24442, 36663, 48884, 61116, 73337, 85558, and 97779.

FOLLOW UP: Which of the numbers defined in the original problem are multiples of 44? [48884 & 61116]

4E **METHOD 1** *Strategy: Make a strategic substitution.*

Substitute $(2000 + 19)^2$ for 2019^2. This implies that $\dfrac{2019^2-2000^2-19^2}{2000\times 19}=\dfrac{(2000+19)^2-2000^2-19^2}{2000\times 19}$.

Notice that the numerator is in the form of the given expansion rewritten as $(a+b)^2-a^2-b^2=2ab$, so

$$\frac{(2000+19)^2-2000^2-19^2}{2000\times 19}=\frac{2\times 2000\times 19}{2000\times 19}=\mathbf{2}.$$

METHOD 2 *Strategy: Apply the distributive property.*

Apply the distributive property in the numerator: $\dfrac{2019^2-2000^2-19^2}{2000\times 19}=\dfrac{2019^2-\left(2000^2+19^2\right)}{2000\times 19}$. Since $(a+b)^2=a^2+2ab+b^2$ we can rewrite it as $a^2+b^2=(a+b)^2-2ab$ and apply this to the numerator to get

$$\frac{2019^2-\left(2000^2+19^2\right)}{2000\times 19}=\frac{2019^2-\left((2000+19)^2-2\times 2000\times 19\right)}{2000\times 19}=\frac{2\times 2000\times 19}{2000\times 19}=2.$$

NOTE: Other FOLLOW-UP problems related to some of the above can be found in our three contest problem books and in "Creative Problem Solving in School Mathematics." Visit www.moems.org for details and to order.

SOLUTIONS AND ANSWERS

5A	**7**
5B	**–1/7**
5C	**22**
5D	**13π**
5E	**0**

5A **METHOD 1** *Strategy: Use approximations to find the nearest perfect square.*

Since 0.499 ≈ 0.5 and 17.1 ≈ 17, $\sqrt{\frac{33}{0.499}-17.1} \approx \sqrt{\frac{33}{0.5}-17} = \sqrt{66-17} = \sqrt{49}$. A good approximation for the given expression is **7**.

METHOD 2 *Strategy: Simplify by finding a common denominator.*
Use 0.499 ≈ 0.5 and 17.1 ≈ 17.

Then $\sqrt{\frac{33}{0.5}-17} = \sqrt{\frac{33}{0.5}-\frac{8.5}{0.5}} = \sqrt{\frac{24.5}{0.5}} = \sqrt{\frac{24.5}{0.5}\times\frac{2}{2}} = \sqrt{49} = 7$.

FOLLOW UPS: (1) Find the integer closest in value to $\sqrt{17.4-\frac{0.499}{0.333}}$. [4]

(2) Find the integer closest in value to $\sqrt[3]{\frac{33}{0.499}-17.1}$. [4]

5B **METHOD 1** *Strategy: Use properties of division and of negative numbers.*
The greatest negative quotient is the negative number closest to zero. By inspection, the negative number closest to zero is (1) ÷ (–7) = **–1/7**.

METHOD 2 *Strategy: Look for a pattern of various quotients.*
To obtain a negative quotient, either the divisor or dividend must be negative, and the other positive. Look for a pattern: When –7 is divided successively by 1, 2, 6, the quotients get larger (closer to zero). The largest of these is –7/6. Similarly, use –5, –4, –3, and –2 as dividends. The largest of all the quotients will be $\frac{-2}{6} = -\frac{1}{3}$. Now successively compare, in the same way, when 0, 1, 2, and 6 are divided by each of –7, –5, –4, –3, and –2. The largest of these is $\frac{1}{-7} = -\frac{1}{7}$. Since $-\frac{1}{7}$ is greater than $-\frac{1}{3}$, the greatest negative quotient is $-\frac{1}{7}$.

FOLLOW UPS: (1) Use the same set of numbers to determine the greatest positive quotient. [6] *(2) Find the least number when two of the numbers from the set are subtracted.* [–13] *(3) When two different numbers are selected from the original set, what is the probability that their product will be greater than 1?* [13/36]

5C *Strategy: Use addition property of zero to maximize the number of integers.*
Notice that –11 + (–10) + (–9) + … + (–1) + 0 + 1 + … + 9 + 10 = –11. Therefore, the greatest number of consecutive integers that sum to –11 is **22**.

FOLLOW UPS: (1) Find the least number of consecutive integers that sum to –11. [2]
(2) Find the number of sets of two or more consecutive integers that sum to –11. [3]

5D **METHOD 1** *Strategy: Use the formula for the area of a circle.*
The area of the larger circle is $A = \pi r^2 = \pi(4)^2 = 16\pi$. Subtract the area of the crosshatched region to find the left portion of the large circle: $16\pi - 6\pi = 10\pi$. Use a similar approach to find the right portion of the smaller circle: $9\pi - 6\pi = 3\pi$. The sum of these two areas is $10\pi + 3\pi =$ $\mathbf{13\pi}$.

METHOD 2 *Strategy: Apply the principle of inclusion-exclusion.*
The sum of the areas of the two regions without crosshatching equals the sum of the areas of the two circles minus twice the area of the overlapped (crosshatched) region: $16\pi + 9\pi - 2(6\pi) = 13\pi$. Notice that when adding the areas of the 2 circles, the crosshatched intersection is counted twice. That is the reason for subtracting it from each of the areas of the circles.

FOLLOW UP: If a dart is thrown at the two circles and hits in the interior of the circles, what is the probability that it will hit in the cross-hatched region? [6/19]

5E **METHOD 1** *Strategy: Solve each part separately.*
1st inequality $2 < |N|$ is true for integers 3, -3, 4, -4, 5, -5, 6, -6, …
2nd inequality $|N| < 6$ is true for integers 5, -5, 4, -4, 3, -3, 2, -2, 1, -1, and 0. The integers that solve both the first inequality and the second inequality are 3, -3, 4, -4, 5, and -5.
Finally, the sum 3 + (-3) + 4 + (-4) + 5 + (-5) = **0**.

METHOD 2 *Strategy: Use substitution.*
Observe that $|x| = |-x|$ for all integers. For example both |8| and |-8| equal 8.

$\lvert N\rvert$	$2 < \lvert N\rvert$	$\lvert N\rvert < 6$	Solutions
0	No	Yes	None
1	No	Yes	None
2	No	Yes	None
3	Yes	Yes	3, -3
4	Yes	Yes	4, -4
5	Yes	Yes	5, -5
6	Yes	No	None

There are no other solutions to both inequalities.
The sum of 3, -3, 4, -4, 5, and -5 is 0.